

## The tableau method for predicate logic

We present a simplified proof system for first order logic that will allow us to prove inconsistency, but not consistency.

More precisely, if  $A$  is a finite set of sentences, then

- if  $A$  inconsistent, i.e., does not have a model, we will get a proof of this fact
- if  $A$  has a model, the method gives us information.

The method generates a tableau that we will be able to close in case  $A$  is inconsistent, but that at any time we will not be able to say if it will close.

## Theorem (Alonzo Church '36)

There is a computer program taking as inputs finite sets of sentences  $A$  etc. The program terminates if and only if  $A$  is inconsistent.

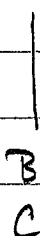
On the other hand, there is no computer program that terminates if and only if  $A$  is consistent.

### The derivation rules

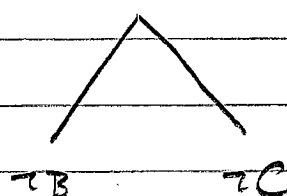
$$(1) \neg\neg B$$



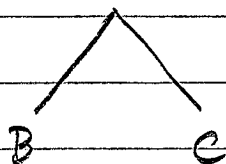
$$(2) B \wedge C$$



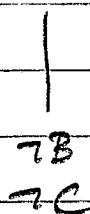
$$(3) \neg(B \wedge C)$$



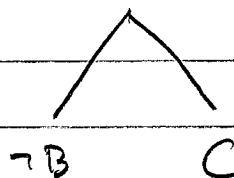
$$(4) B \vee C$$



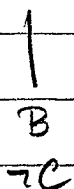
$$(5) \neg(B \vee C)$$



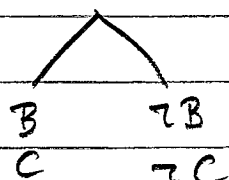
$$(6) B \rightarrow C$$



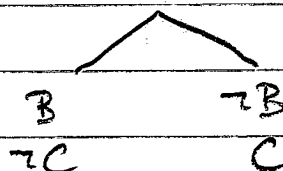
$$(7) \neg(B \rightarrow C)$$

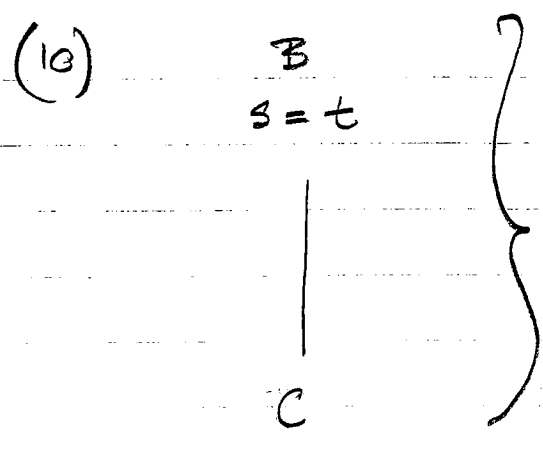


$$(8) B \leftrightarrow C$$

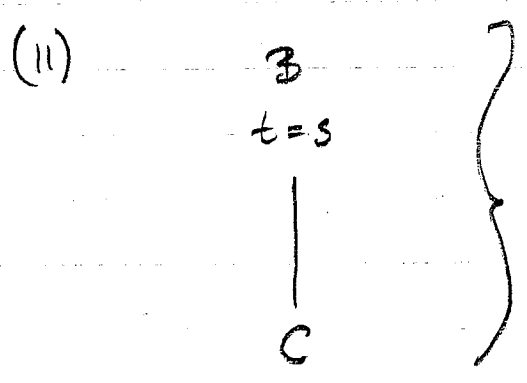


$$(9) \neg(B \leftrightarrow C)$$

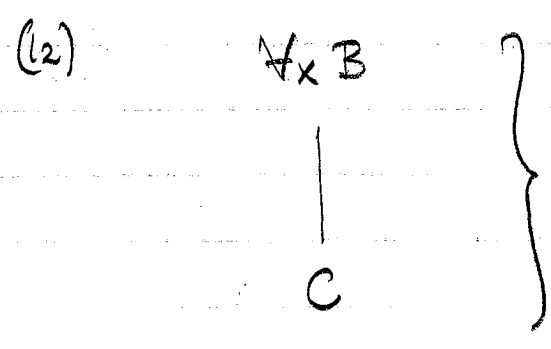




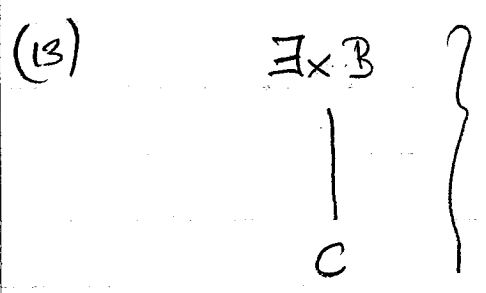
where  $C$  is obtained from  $B$  by replacing one or more occurrences of the term  $s$  by the term  $t$



\_\_\_\_\_  $\alpha$  \_\_\_\_\_



where  $C$  is obtained from  $B$  by replacing each free occurrence of the variable  $x$  in  $B$  by some term  $t$  that occurs in a formula in the branch above " $\forall x B$ "



where  $C$  is obtained from  $B$  by replacing each free occurrence of the variable  $x$  in  $B$  by a new constant term not occurring in any formula in the branch above " $\exists x B$ ".

(14)  $\neg \forall x B$



$\exists x \neg B$

(15)  $\neg \exists x B$



$\forall x \neg B$

Note We do not check off formulas that have been used since these can be reused at a later point. Also, by Church's theorem, we cannot conclude consistency simply by not being able to close the tableau.

We close a branch whenever we find a formula  $B$  and its negation  $\neg B$  in the same branch. Similarly, if we find a formula " $\neg S = S$ ".

Example Show that

$$A = \{ \forall x ffx = x, \forall x fffx = x, \neg \forall x x = fx \}$$

is an inconsistent set of sentences, by showing that its tableau closes.

$$\forall x \quad ffx = x$$

$$\forall x \quad fffx = x$$

$$\neg \forall x \quad x = fx$$

$$\begin{array}{c} | \\ \exists x \quad \neg x = fx \end{array}$$

$$(*) \quad \neg a = fa$$

$$ffa = a$$

$$fffa = a$$

where  $a$  is a new constant symbol.

$$(**) \quad \neg a = fffa$$

$$\neg fffa = fffa$$


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replacing the right hand occurrence of  $a$  in  $(*)$  by  $ffa$

replacing the left hand occurrence of  $a$  in  $(**)$  by  $fffa$ .

Example Show that the following argument is valid:

$$\frac{\exists x (P_x \wedge \forall y (B_y \rightarrow G_{xy}))}{\forall y (B_y \rightarrow \exists x (P_x \wedge G_{xy}))}$$

$$\forall y (B_y \rightarrow \exists x (P_x \wedge G_{xy}))$$

$$\exists x (P_x \wedge \forall y (B_y \rightarrow G_{xy}))$$

$$\neg \forall y (B_y \rightarrow \exists x (P_x \wedge G_{xy}))$$

new constant a :  $P_a \wedge \forall y (B_y \rightarrow G_{ay})$

$$\exists y \neg (B_y \rightarrow \exists x (P_x \wedge G_{xy}))$$

new constant b :  $\neg (B_b \rightarrow \exists x (P_x \wedge G_{xb}))$

$$\begin{array}{c} B_b \\ \neg \exists x (P_x \wedge G_{xb}) \end{array}$$

$$\forall x \neg (P_x \wedge G_{xb})$$

$$\neg (P_a \wedge G_{ab})$$

$$P_a$$

$$\forall y (B_y \rightarrow G_{ay})$$

$$B_b \rightarrow G_{ab}$$

$$\underline{\neg B_b}$$

$$G_{ab}$$

$$G_{ab}$$

$$\underline{P_a}$$

$$\underline{\neg G_{ab}}$$